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On Dual Hybrid Mersenne and Mersenne-Lucas Sequences

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Abstract

In this paper, we introduce Dual Hybrid Mersenne and Mersenne-Lucas sequences and establish recurrence relations, generating functions, and Binet formulas for the preceding sequence. Also verified the above sequences through some widely acknowledged identities and furthered a few relationships among them.

Keywords: Binet formula, Mersenne sequence, Hybrid sequence, Catalan identity, Dual sequence

1. Introduction

The most well-known series of positive integers is the Fibonacci sequence. The Lucas, Pell, and Jacobsthal sequences are only a few of the fundamental structures that can be made using this series under various initial conditions[5-6]. Marin Mersenne, a French mathematician, initially introduced a number of the type $\mathfrak{M}_n = 2^n - 1$, where n is an integer, in 1644. Various investigations have been carried out on the Mersenne sequences. Mersenne-Lucas sequences are defined as $\mathfrak{M}L_n = 2^n + 1, n \geq 2$, with $\mathfrak{M}L_0 = 2, \mathfrak{M}L_1 = 3$. In [1-3], the Mersenne-Lucas sequences, including its generating functions and Binet formulas were discussed. To know more about the dual sequence, see [4]. In this paper, dual numbers of the Mersenne and Mersenne-Lucas sequences are combined with hybrid number system and Dual hybrid Mersenne and Mersenne-Lucas sequences are produced. The recurrence relations, generating functions, and Binet formulas for the preceding sequence are established. Also the above sequences are verified through some widely acknowledged identities and furthered a few relationships among them.

2. Main Definitions and Results

Definition 2.1

The Dual Hybrid Mersenne sequence is defined for any $n \geq 0$ by

$$\mathbb{M}_n = \mathfrak{M}H_n + \mathfrak{M}H_{n+1}\varepsilon, \tag{2.1}$$

where $\mathfrak{M}H_n = \mathfrak{M}_n + \mathfrak{M}_{n+1}i + \mathfrak{M}_{n+2}\varepsilon + \mathfrak{M}_{n+3}h$ is the n^{th} Mersenne hybrid number.

Definition 2.2

The Dual Hybrid Mersenne-Lucas sequences are defined for any positive integer n, by

$$\mathbb{M}\mathrm{L_n} = \mathfrak{M}\mathrm{LH_n} + \mathfrak{M}\mathrm{LH_{n+1}}\varepsilon, \tag{2.2}$$

where $\mathfrak{M}\mathrm{LH_n} = \mathfrak{M}\mathrm{L_n} + \mathfrak{M}\mathrm{L_{n+1}}\mathrm{i} + \mathfrak{M}\mathrm{L_{n+2}}\varepsilon + \mathfrak{M}\mathrm{L_{n+3}}\mathrm{h}$ is the n^{th} Mersenne-Lucas hybrid number.

Let $\mathbb{M}_n = \mathcal{M}H_n + \mathcal{M}H_{n+1}\varepsilon$ and $\mathbb{M}_m = \mathcal{M}H_m + \mathcal{M}H_{m+1}\varepsilon$ be two Dual hybrid Mersenne numbers and A=$A_1 + A_2\varepsilon$ be a Dual number. Then, the arithmetic operations are defined as follows:

$$\mathbb{M}_n \pm \mathbb{M}_m = (\mathcal{M}H_n \pm \mathcal{M}H_m) + (\mathcal{M}H_{n+1} \pm \mathcal{M}H_{m+1})\varepsilon$$

$$A\mathbb{M}_n = A_1\mathcal{M}H_n + (A_1\mathcal{M}H_{n+1} + A_2\mathcal{M}H_n)\varepsilon$$

$$\mathbb{M}_n\mathbb{M}_m = \mathcal{M}H_n\mathcal{M}H_m + (\mathcal{M}H_n\mathcal{M}H_{m+1} + \mathcal{M}H_{n+1}\mathcal{M}H_m)\varepsilon$$

Theorem 2.1

The Dual Hybrid Mersenne and Mersenne-Lucas numbers satisfy the recurrence relations

(i) $\mathbb{M}_{n+1} = 3\mathbb{M}_n - 2\mathbb{M}_{n-1}$ (2.3)

(ii) $\mathbb{M}L_{n+1} = 3\mathbb{M}L_n - 2\mathbb{M}L_{n-1}$ (2.4)

Proof

(i)
$$\begin{aligned}
3\mathbb{M}_n - 2\mathbb{M}_{n-1} &= 3(\mathcal{M}H_n + \mathcal{M}H_{n+1}\varepsilon) - 2(\mathcal{M}H_{n-1} + \mathcal{M}H_n\varepsilon) \\
&= 3(\mathcal{M}_n + \mathcal{M}_{n+1}i + \mathcal{M}_{n+2}\varepsilon + \mathcal{M}_{n+3}h + (\mathcal{M}_{n+1} + \mathcal{M}_{n+2}i + \mathcal{M}_{n+3}\varepsilon + \mathcal{M}_{n+4}h)\varepsilon) \\
&\quad - 2(\mathcal{M}_{n-1} + \mathcal{M}_n i + \mathcal{M}_{n+1}\varepsilon + \mathcal{M}_{n+2}h \\
&\quad + (\mathcal{M}_n + \mathcal{M}_{n+1}i + \mathcal{M}_{n+2}\varepsilon + \mathcal{M}_{n+3}h)\varepsilon) \\
&= 3\mathcal{M}_n - 2\mathcal{M}_{n-1} + (3\mathcal{M}_{n+1} - 2\mathcal{M}_n)i + (3\mathcal{M}_{n+2} - 2\mathcal{M}_{n+1})\varepsilon + 3(\mathcal{M}_{n+3} - 2\mathcal{M}_{n+2})h \\
&\quad + [3\mathcal{M}_{n+1} - 2\mathcal{M}_n + (3\mathcal{M}_{n+2} - 2\mathcal{M}_{n+1})i + (3\mathcal{M}_{n+3} - 2\mathcal{M}_{n+2})\varepsilon \\
&\quad + (3\mathcal{M}_{n+4} - 2\mathcal{M}_{n+3})h]\varepsilon \\
&= \mathcal{M}_{n+1} + \mathcal{M}_{n+2}i + \mathcal{M}_{n+3}\varepsilon + \mathcal{M}_{n+4}h + [\mathcal{M}_{n+2} + \mathcal{M}_{n+3}i + \mathcal{M}_{n+4}\varepsilon + \mathcal{M}_{n+5}h]\varepsilon \\
&= \mathcal{M}H_{n+1} + \mathcal{M}H_{n+2}\varepsilon \\
&= \mathbb{M}_{n+1}
\end{aligned}$$

(ii) $3\mathbb{M}L_n - 2\mathbb{M}L_{n-1} = 3(\mathcal{M}LH_n + \mathcal{M}LH_{n+1}\varepsilon) - 2(\mathcal{M}LH_{n-1} + \mathcal{M}LH_n\varepsilon)$

$$= 3(\mathcal{ML}_n + \mathcal{ML}_{n+1}i + \mathcal{ML}_{n+2}\varepsilon + \mathcal{ML}_{n+3}h + (\mathcal{ML}_{n+1} + \mathcal{ML}_{n+2}i + \mathcal{ML}_{n+3}\varepsilon + \mathcal{ML}_{n+4}h)\varepsilon) - 2(\mathcal{ML}_{n-1} + \mathcal{ML}_n i + \mathcal{ML}_{n+1}\varepsilon + \mathcal{ML}_{n+2}h + (\mathcal{ML}_n + \mathcal{ML}_{n+1}i + \mathcal{ML}_{n+2}\varepsilon + \mathcal{ML}_{n+3}h)\varepsilon)$$

$$= 3\mathcal{ML}_n - 2\mathcal{ML}_{n-1} + (3\mathcal{ML}_{n+1} - 2\mathcal{ML}_n)i + (3\mathcal{ML}_{n+2} - 2\mathcal{ML}_{n+1})\varepsilon + (3\mathcal{ML}_{n+3} - 2\mathcal{ML}_{n+2})h + [3\mathcal{ML}_{n+1} - 2\mathcal{ML}_n + (3\mathcal{ML}_{n+2} - 2\mathcal{ML}_{n+1})i + (3\mathcal{ML}_{n+3} - 2\mathcal{ML}_{n+2})\varepsilon + (3\mathcal{ML}_{n+4} - 2\mathcal{ML}_{n+3})h]\varepsilon$$

$$= \mathcal{ML}_{n+1} + \mathcal{ML}_{n+2}i + \mathcal{ML}_{n+3}\varepsilon + \mathcal{ML}_{n+4}h + [\mathcal{ML}_{n+2} + \mathcal{ML}_{n+3}i + \mathcal{ML}_{n+4}\varepsilon + \mathcal{ML}_{n+5}h]\varepsilon$$

$$= \mathcal{MLH}_{n+1} + \mathcal{MLH}_{n+2}\varepsilon$$

$$= \mathbb{ML}_{n+1}$$

Theorem 2.2

The Binet formulae for dual hybrid Mersenne and Mersenne-Lucas sequences are

(i) $\mathbb{M}_n = 2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*$ (2.5)

(ii) $\mathbb{M}_n = 2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*$ (2.6)

where $\mathcal{A} = 1 + 2i + 2^2\varepsilon + 2^3h, \mathcal{B} = 1 + i + \varepsilon + h, \lambda^* = 1 + 2\varepsilon$ and $\mu^* = 1 + \varepsilon$

Proof

By using the binet formulae for Mersenne and Mersenne-Lucas sequences, we obtain

(i) $\mathbb{M}_n = \mathfrak{M}H_n + \mathfrak{M}H_{n+1}\varepsilon$

$$= \mathfrak{M}_n + \mathfrak{M}_{n+1}i + \mathfrak{M}_{n+2}\varepsilon + \mathfrak{M}_{n+3}h + (\mathfrak{M}_{n+1} + \mathfrak{M}_{n+2}i + \mathfrak{M}_{n+3}\varepsilon + \mathfrak{M}_{n+4}h)\varepsilon$$

$$= (2^n - 1) + (2^{n+1} - 1)i + (2^{n+2} - 1)\varepsilon + (2^{n+3} - 1)h + [(2^{n+1} - 1) + (2^{n+2} - 1)i + (2^{n+3} - 1)\varepsilon + (2^{n+4} - 1)h]\varepsilon$$

$$= 2^n(1 + 2i + 2^2\varepsilon + 2^3h) - (1 + i + \varepsilon + h) + [2^{n+1}(1 + 2i + 2^2\varepsilon + 2^3h) - (1 + i + \varepsilon + h)]\varepsilon$$

$$= 2^n(1 + 2i + 2^2\varepsilon + 2^3h)(1 + 2\varepsilon) - (1 + i + \varepsilon + h)(1 + \varepsilon)$$

$$= 2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*$$

(ii) $\mathbb{ML}_n = \mathcal{MLH}_n + \mathcal{MLH}_{n+1}\epsilon$

$$= \mathcal{ML}_n + \mathcal{ML}_{n+1}i + \mathcal{ML}_{n+2}\epsilon + \mathcal{ML}_{n+3}h + (\mathcal{ML}_{n+1} + \mathcal{ML}_{n+2}i + \mathcal{ML}_{n+3}\epsilon + \mathcal{ML}_{n+4}h)\epsilon$$

$$= (2^n + 1) + (2^{n+1} + 1)i + (2^{n+2} + 1)\epsilon + (2^{n+3} + 1)h + [(2^{n+1} + 1) + (2^{n+2} + 1)i + (2^{n+3} + 1)\epsilon + (2^{n+4} + 1)h]\epsilon$$

$$= 2^n(1 + 2i + 2^2\epsilon + 2^3h) + (1 + i + \epsilon + h) + [2^{n+1}(1 + 2i + 2^2\epsilon + 2^3h) + (1 + i + \epsilon + h)]\epsilon$$

$$= 2^n(1 + 2i + 2^2\epsilon + 2^3h)(1 + 2\epsilon) + (1 + i + \epsilon + h)(1 + \epsilon)$$

$$= 2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*$$

Theorem 2.3

The generating function for the dual hybrid Mersenne and Mersenne-Lucas sequences are

(i) $f_{\mathbb{M}_n}(x) = \frac{\mathbb{M}_0 + (\mathbb{M}_1 - 3\mathbb{M}_0)x}{1-3t+2t^2}$

(ii) $g_{\mathbb{ML}_n}(x) = \frac{\mathbb{ML}_0 + (\mathbb{ML}_1 - 3\mathbb{ML}_0)x}{1-3t+2t^2}$

Proof

(i) Since,

$$f_{\mathbb{M}_n}(x) = \sum_{i=0}^{\infty} \mathbb{M}_i\, x^i = \mathbb{M}_0 + \mathbb{M}_1 x + \sum_{i=2}^{\infty} \mathbb{M}_i\, x^i \tag{2.7}$$

Multiplying (2.7) by $-3x$ and $2x^2$ respectively, we get

$$-3xf_{\mathbb{M}_n}(x) = -3x\sum_{i=0}^{\infty} \mathbb{M}_i\, x^i = -3\mathbb{M}_0 x - 3\sum_{i=2}^{\infty} \mathbb{M}_{i-1}\, x^i, \tag{2.8}$$

$$2x^2 f_{\mathbb{M}_n}(x) = 2x^2\sum_{i=0}^{\infty} \mathbb{M}_i\, x^i = 2\sum_{i=2}^{\infty} \mathbb{M}_{i-2}\, x^i \tag{2.9}$$

Adding (2.7-2.9), we obtain

$$(1 - 3x + 2x^2)f_{\mathbb{M}_n}(x) = \mathbb{M}_0 + \mathbb{M}_1 x + \sum_{i=2}^{\infty} \mathbb{M}_i\, x^i - 3\mathbb{M}_0 x - 3\sum_{i=2}^{\infty} \mathbb{M}_{i-1}\, x^i + 2\sum_{i=2}^{\infty} \mathbb{M}_{i-2}\, x^i$$

$$= \mathbb{M}_0 + x(\mathbb{M}_1 - 3\mathbb{M}_0) + \sum_{i=2}^{\infty}[\mathbb{M}_i - 3\mathbb{M}_{i-1} + 2\mathbb{M}_{i-2}]\, x^i. \tag{2.10}$$

Hence, $f_{\mathbb{M}_n}(x) = \frac{\mathbb{M}_0 + (\mathbb{M}_1 - 3\mathbb{M}_0)x}{1-3t+2t^2}$.

(ii) Since,

$$g_{\mathbb{ML}_n}(x) = \sum_{i=0}^{\infty} \mathbb{ML}_i\, x^i \tag{2.11}$$

$$= \mathbb{ML}_0 + \mathbb{ML}_1 x + \sum_{i=2}^{\infty} \mathbb{ML}_i\, x^i$$

Multiplying (2.11) by $-3x$ and $2x^2$ respectively, we get

$$-3xg_{\mathbb{ML}_n}(t) = -3x\sum_{i=0}^{\infty}\mathbb{ML}_i\, x^i = -3\mathbb{ML}_0 x - 3\sum_{i=2}^{\infty}\mathbb{ML}_{i-1}\, x^i, \tag{2.12}$$

$$2x^2 g_{\mathbb{ML}_n}(t) = 2x^2\sum_{i=0}^{\infty}\mathbb{ML}_i\, x^i = 2\sum_{i=2}^{\infty}\mathbb{ML}_{i-2}\, x^i. \tag{2.13}$$

Adding the equations (2.11-2.13), we obtain

$$\begin{aligned}(1-3x+2x^2)g_{\mathbb{ML}_n}(t) &= \mathbb{ML}_0 + \mathbb{ML}_1 x + \sum_{i=2}^{\infty}\mathbb{ML}_i\, x^i + -3\mathbb{ML}_0 x - 3\sum_{i=2}^{\infty}\mathbb{ML}_{i-1}\, x^i \\ &+ 2\sum_{i=2}^{\infty}\mathbb{ML}_{i-2}\, x^i \\ &= \mathbb{ML}_0 + x(\mathbb{ML}_1 - 3\mathbb{ML}_0) \\ &+ \sum_{i=2}^{\infty}[\mathbb{ML}_i - 3\mathbb{ML}_{i-1} + 2\mathbb{ML}_{i-2}]\, x^i \\ &= \mathbb{ML}_0 + (\mathbb{ML}_1 - 3\mathbb{ML}_0)\end{aligned}$$

Hence, $g_{\mathbb{ML}_n}(t) = \frac{\mathbb{ML}_0 + (\mathbb{ML}_1 - 3\mathbb{ML}_0)t}{1-3t+2t^2}$.

Theorem 2.4

The exponential generating function for the dual hybrid Mersenne and Mersenne-Lucas sequences are

(i) $\sum_{k=0}^{\infty}\frac{\mathbb{M}_k l^k}{k!} = \mathcal{A}\lambda^* e^{2l} - \mathcal{B}\mu^* e^l$

(ii) $\sum_{k=0}^{\infty}\frac{\mathbb{ML}_k l^k}{k!} = \mathcal{A}\lambda^* e^{2l} + \mathcal{B}\mu^* e^l$

Proof

(i) $\sum_{k=0}^{\infty}\frac{\mathbb{M}_k l^k}{k!} = \sum_{k=0}^{\infty}(2^k\mathcal{A}\,\lambda^* - \mathcal{B}\mu^*)\frac{l^k}{k!}$

$$= \sum_{k=0}^{\infty}\frac{(2l)^k}{k!}\mathcal{A}\,\lambda^* - \sum_{k=0}^{\infty}\frac{l^k}{k!}\mathcal{B}\mu^*$$

$$\sum_{k=0}^{\infty}\frac{\mathbb{M}_k l^k}{k!} = \mathcal{A}\lambda^* e^{2l} - \mathcal{B}\mu^* e^l$$

(ii) $\sum_{k=0}^{\infty}\frac{\mathbb{ML}_k l^k}{k!} = \sum_{k=0}^{\infty}(2^k\mathcal{A}\,\lambda^* + \mathcal{B}\mu^*)\frac{l^k}{k!}$

$$= \sum_{k=0}^{\infty}\frac{(2l)^k}{k!}\mathcal{A}\,\lambda^* + \sum_{k=0}^{\infty}\frac{l^k}{k!}\mathcal{B}\mu^*$$

$$= \mathcal{A}\lambda^* e^{2l} + \mathcal{B}\mu^* e^l$$

Theorem 2.5

For any positive integer n, we have

(i) $\mathbb{M}_n + \mathbb{ML}_n = 2^{n+1}\mathcal{A}\lambda^*$

(ii) $\mathbb{M}_n - \mathbb{ML}_n = -2\mathcal{B}\mu^*$

Proof

(i) $\mathbb{M}_n + \mathbb{ML}_n = 2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^* + 2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*$[From (2.5)&(2.6)]

$= 2^{n+1}\mathcal{A}\lambda^*$

(ii) $\mathbb{M}_n - \mathbb{ML}_n = 2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^* - 2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*$

$= -2\mathcal{B}\mu^*$

Theorem 2.6

For any positive integer n, we have

(i) $\mathbb{M}_n + \mathbb{M}_{n+1} = 3(2^n)\mathcal{A}\lambda^* - 2\mathcal{B}\mu^*$

(ii) $\mathbb{ML}_n + \mathbb{ML}_{n+1} = 3(2^n)\mathcal{A}\lambda^* + 2\mathcal{B}\mu^*$

Proof

(i) $\mathbb{M}_n + \mathbb{M}_{n+1} = 2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^* + 2^{n+1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*$

$= 3(2^n)\mathcal{A}\lambda^* - 2\mathcal{B}\mu^*.$

(ii) $\mathbb{ML}_n + \mathbb{ML}_{n+1} = 2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^* + 2^{n+1}\mathcal{A}\lambda^* + \mathcal{B}\mu^*$

$= 3(2^n)\mathcal{A}\lambda^* + 2\mathcal{B}\mu^*.$

Theorem 2.7

Let m, n, r &s each be positive integers, then

(i) $\mathbb{M}_m\mathbb{ML}_n + \mathbb{ML}_m\mathbb{M}_n = 2\left(2^{m+n}\mathcal{A}^2\lambda^{*2} - \mathcal{B}^2\mu^{*2}\right)$,

(ii) $\mathbb{M}_m\mathbb{ML}_n - \mathbb{ML}_m\mathbb{M}_n = 2(2^m\mathcal{A}\lambda^*\mathcal{B}\mu^* - 2^n\mathcal{B}\mu^*\mathcal{A}\lambda^*)$, for $n \geq 0, m \geq 0$,

(iii) $\mathbb{M}_{n+r}\mathbb{ML}_{n+s} - \mathbb{M}_{n+s}\mathbb{ML}_{n+r} = 2^n(\mathfrak{M}_r - \mathfrak{M}_s)(\mathcal{A}\lambda^*\mathcal{B}\mu^* + \mathcal{B}\mu^*\mathcal{A}\lambda^*)$.

Proof

(i) $\mathbb{M}_m\mathbb{ML}_n + \mathbb{ML}_m\mathbb{M}_n$

$= (2^m\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*) + (2^m\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*)$

$= 2\left(2^{m+n}\mathcal{A}^2\lambda^{*2} - \mathcal{B}^2\mu^{*2}\right)$

(ii) $\mathbb{M}_m \mathbb{M}\mathrm{L}_n - \mathbb{M}\mathrm{L}_m \mathbb{M}_n$

$$= (2^m \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^n \mathcal{A}\lambda^* + \mathcal{B}\mu^*) + (2^m \mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^n \mathcal{A}\lambda^* - \mathcal{B}\mu^*)$$

$$= 2(2^{\mathrm{m}} \mathcal{A}\lambda^* \mathcal{B}\mu^* - 2^{\mathrm{n}} \mathcal{B}\mu^* \mathcal{A}\lambda^*).$$

(iii) $\mathbb{M}_{n+r} \mathbb{M}\mathrm{L}_{n+s} - \mathbb{M}_{n+s} \mathbb{M}\mathrm{L}_{n+r}$

$$= (2^{\mathrm{n+r}} \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{\mathrm{n+s}} \mathcal{A}\lambda^* + \mathcal{B}\mu^*)$$

$$-(2^{\mathrm{n+s}} \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{\mathrm{n+r}} \mathcal{A}\lambda^* + \mathcal{B}\mu^*)$$

$$= 2^{\mathrm{n}}(\mathfrak{M}_{\mathrm{r}} - \mathfrak{M}_{\mathrm{s}})(\mathcal{A}\lambda^* \mathcal{B}\mu^* + \mathcal{B}\mu^* \mathcal{A}\lambda^*).$$

Theorem 2.8 Vajda Identity

For any non-negative integers $m, n$and r we have

(i) $\mathbb{M}_{n+r} \mathbb{M}_{n+s} - \mathbb{M}_n \mathbb{M}_{n+r+s} = \lambda^* \mu^* 2^n \mathcal{M}_r (2^s \mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})$

(ii) $\mathbb{M}L_{n+r} \mathbb{M}L_{n+s} - \mathbb{M}L_n \mathbb{M}L_{n+r+s} = \lambda^* \mu^* 2^n \mathcal{M}_r (\mathcal{A}\mathcal{B} - 2^s \mathcal{B}\mathcal{A})$

Proof

(i) $\mathbb{M}_{n+r} \mathbb{M}_{n+s} - \mathbb{M}_n \mathbb{M}_{n+r+s}$

$$= (2^{n+r} \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{n+s} \mathcal{A}\lambda^* - \mathcal{B}\mu^*) - (2^n \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{n+r+s} \mathcal{A}\lambda^* - \mathcal{B}\mu^*)$$

$$= \mathcal{A}\mathcal{B}\lambda^* \mu^* 2^n (1 - 2^r) + \mathcal{B}\mathcal{A}\mu^* \lambda^* 2^n (2^{r+s} - 2^s)$$

$$= -\mathcal{A}\mathcal{B}\lambda^* \mu^* 2^n \mathcal{M}_r + \mathcal{B}\mathcal{A}\lambda^* \mu^* 2^{n+s} \mathcal{M}_r$$

$$= \lambda^* \mu^* 2^n \mathcal{M}_r (2^s \mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})$$

(ii) $\mathbb{M}L_{n+r} \mathbb{M}L_{n+s} - \mathbb{M}L_n \mathbb{M}L_{n+r+s}$

$$= (2^{n+r} \mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^{n+s} \mathcal{A}\lambda^* + \mathcal{B}\mu^*) - (2^n \mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^{n+r+s} \mathcal{A}\lambda^* + \mathcal{B}\mu^*)$$

$$= \mathcal{A}\mathcal{B}\lambda^* \mu^* 2^n \mathcal{M}_r - \mathcal{B}\mathcal{A}\lambda^* \mu^* 2^{n+s} \mathcal{M}_r$$

$$= \lambda^* \mu^* 2^n \mathcal{M}_r (\mathcal{A}\mathcal{B} - 2^s \mathcal{B}\mathcal{A})$$

Theorem 2.9 Catalan Identity

For any non-negative integers $n$and s such that $n \geq s$, we have

(i) $\mathbb{M}_{n-s} \mathbb{M}_{n+s} - \mathbb{M}_n^2 = \lambda^* \mu^* \mathcal{M}_s 2^{n-s} (\mathcal{A}\mathcal{B} - 2^s \mathcal{B}\mathcal{A})$

(ii) $\mathbb{M}L_{n-s} \mathbb{M}L_{n+s} - \mathbb{M}L_n^2 = \lambda^* \mu^* \mathcal{M}_s 2^{n-s} (2^s \mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})$

Proof

(i) $\mathbb{M}_{n-s} \mathbb{M}_{n+s} - \mathbb{M}_n^2$

$$= (2^{n-s} \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{n+s} \mathcal{A}\lambda^* - \mathcal{B}\mu^*) - (2^n \mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^n \mathcal{A}\lambda^* - \mathcal{B}\mu^*)$$

$$= \mathcal{A}\mathcal{B}\lambda^*\mu^*(-2^{n-s} + 2^n) + \mathcal{B}\mathcal{A}\mu^*\lambda^*(-2^{n+s} + 2^n)$$
$$= \mathcal{A}\mathcal{B}\lambda^*\mu^*\mathcal{M}_s 2^{n-s} - \mathcal{B}\mathcal{A}\mu^*\lambda^*\mathcal{M}_s 2^n$$
$$= \lambda^*\mu^*\mathcal{M}_s 2^{n-s}(\mathcal{A}\mathcal{B} - 2^s\mathcal{B}\mathcal{A})$$

(ii) $\mathbb{M}L_{n-s}\mathbb{M}L_{n+s} - \mathbb{M}L_n^2$

$$= (2^{n-s}\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^{n+s}\mathcal{A}\lambda^* + \mathcal{B}\mu^*) - (2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*)$$
$$= \mathcal{A}\mathcal{B}\lambda^*\mu^*(2^{n-s} - 2^n) + \mathcal{B}\mathcal{A}\mu^*\lambda^*(2^{n+s} - 2^n)$$
$$= -\mathcal{A}\mathcal{B}\mathcal{M}_s 2^{n-s}\lambda^*\mu^* + \mathcal{B}\mathcal{A}\mu^*\lambda^*\mathcal{M}_s 2^n$$
$$= \lambda^*\mu^*\mathcal{M}_s 2^{n-s}(2^s\mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})$$

Theorem 2.10 Cassini Identity

For any positive integer n,

(i) $\mathbb{M}_{n-1}\mathbb{M}_{n+1} - \mathbb{M}_n^2 = \lambda^*\mu^* 2^{n-1}(\mathcal{A}\mathcal{B} - 2\mathcal{B}\mathcal{A})$

(ii) $\mathbb{M}L_{n-1}\mathbb{M}L_{n+1} - \mathbb{M}L_n^2 = \lambda^*\mu^* 2^{n-1}(2\mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})$

Proof

(i) $\mathbb{M}_{n-1}\mathbb{M}_{n+1} - \mathbb{M}_n^2$

$$= (2^{n-1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{n+1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*) - (2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*)$$
$$= \mathcal{A}\mathcal{B}\lambda^*\mu^*(-2^{n-1} + 2^n) + \mathcal{B}\mathcal{A}\mu^*\lambda^*(-2^{n+1} + 2^n)$$
$$= \lambda^*\mu^* 2^{n-1}(\mathcal{A}\mathcal{B} - 2\mathcal{B}\mathcal{A})$$

(ii) $\mathbb{M}L_{n-1}\mathbb{M}L_{n+1} - \mathbb{M}L_n^2$

$$= (2^{n-1}\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^{n+1}\mathcal{A}\lambda^* + \mathcal{B}\mu^*) - (2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*)$$
$$= \mathcal{A}\mathcal{B}\lambda^*\mu^*(2^{n-1} - 2^n) + \mathcal{B}\mathcal{A}\mu^*\lambda^*(2^{n+1} - 2^n)$$
$$= \lambda^*\mu^* 2^{n-1}(2\mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})$$

Theorem 2.11d'Ocagne Identity

For non-negative integers $m$and n, such that $m \geq n$, we have

(i) $\mathbb{M}_{n+1}\mathbb{M}_m - \mathbb{M}_n\mathbb{M}_{m+1} = \lambda^*\mu^*(2^m\mathcal{B}\mathcal{A} - 2^n\mathcal{A}\mathcal{B})$

(ii) $\mathbb{M}L_{n+1}\mathbb{M}L_m - \mathbb{M}L_n\mathbb{M}L_{m+1} = \lambda^*\mu^*(2^n\mathcal{A}\mathcal{B} - 2^m\mathcal{B}\mathcal{A})$

Proof

(i) $\mathbb{M}_{n+1}\mathbb{M}_m - \mathbb{M}_n\mathbb{M}_{m+1}$

$$= (2^{n+1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^m\mathcal{A}\lambda^* - \mathcal{B}\mu^*) - (2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{m+1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*)$$

$$= \mathcal{AB}\lambda^*\mu^*(-2^{n+1} + 2^n) + \mathcal{BA}\mu^*\lambda^*(-2^m + 2^{m+1})$$
$$= \lambda^*\mu^*(2^m\mathcal{BA} - 2^n\mathcal{AB})$$

(ii) $\mathbb{M}L_{n+1}\mathbb{M}L_m - \mathbb{M}L_n\mathbb{M}L_{m+1}$

$$= (2^{n+1}\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^m\mathcal{A}\lambda^* + \mathcal{B}\mu^*) - (2^n\mathcal{A}\lambda^* + \mathcal{B}\mu^*)(2^{m+1}\mathcal{A}\lambda^* + \mathcal{B}\mu^*)$$
$$= \mathcal{AB}\lambda^*\mu^*(2^{n+1} - 2^n) + \mathcal{BA}\mu^*\lambda^*(2^m - 2^{m+1})$$
$$= \lambda^*\mu^*(2^n\mathcal{AB} - 2^m\mathcal{BA})$$

Theorem 2.12 Honsberger Identity

For any positive integer m and n, we have

(i) $\mathbb{M}_{m-1}\mathbb{M}_n + \mathbb{M}_m\mathbb{M}_{n+1} = 5(\mathcal{A})^2(\lambda^*)^2 2^{m+n-1} - 3\mathcal{AB}\lambda^*\mu^* 2^{m-1} - 3\mathcal{BA}\mu^*\lambda^* 2^n + 2(\mathcal{B})^2(\mu^*)^2$

(ii) $\mathbb{M}L_{m-1}\mathbb{M}L_n + \mathbb{M}L_m\mathbb{M}L_{n+1} = 5(\mathcal{A})^2(\lambda^*)^2 2^{m+n-1} + 3\mathcal{AB}\lambda^*\mu^* 2^{m-1} + 3\mathcal{BA}\mu^*\lambda^* 2^n + 2(\mathcal{B})^2(\mu^*)^2$

Proof

(i) $\mathbb{M}_{m-1}\mathbb{M}_n + \mathbb{M}_m\mathbb{M}_{n+1} = (2^{m-1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^n\mathcal{A}\lambda^* - \mathcal{B}\mu^*) + (2^m\mathcal{A}\lambda^* - \mathcal{B}\mu^*)(2^{n+1}\mathcal{A}\lambda^* - \mathcal{B}\mu^*)$

$$= 2^{m+n-1}(\mathcal{A})^2(\lambda^*)^2 - \mathcal{AB}2^{m-1}\lambda^*\mu^* - \mathcal{BA}2^n\mu^*\lambda^* + (\mathcal{B})^2(\mu^*)^2$$
$$+ 2^{m+n+1}(\mathcal{A})^2(\lambda^*)^2 - \mathcal{AB}2^m\lambda^*\mu^* - \mathcal{BA}2^{n+1}\mu^*\lambda^* + (\mathcal{B})^2(\mu^*)^2$$
$$= 5(\mathcal{A})^2(\lambda^*)^2 2^{m+n-1} - 3\mathcal{AB}\lambda^*\mu^* 2^{m-1} - 3\mathcal{BA}\mu^*\lambda^* 2^n + 2(\mathcal{B})^2(\mu^*)^2$$

(ii) $\mathbb{M}L_{m-1}\mathbb{M}L_n + \mathbb{M}L_m\mathbb{M}L_{n+1} = (2^{m-1}A\lambda^* + \mathcal{B}\mu^*)(2^n A\lambda^* + \mathcal{B}\mu^*) + (2^m A\lambda^* + \mathcal{B}\mu^*)(2^{n+1}A\lambda^* + \mathcal{B}\mu^*)$

$$= 2^{m+n-1}(\mathcal{A})^2(\lambda^*)^2 + \mathcal{AB}2^{m-1}\lambda^*\mu^* + \mathcal{BA}2^n\mu^*\lambda^* + (\mathcal{B})^2(\mu^*)^2 + 2^{m+n+1}(\mathcal{A})^2(\lambda^*)^2$$
$$+ \mathcal{AB}2^m\lambda^*\mu^* + \mathcal{BA}2^{n+1}\mu^*\lambda^* + (\mathcal{B})^2(\mu^*)^2$$
$$= 5(\mathcal{A})^2(\lambda^*)^2 2^{m+n-1} + 3\mathcal{AB}\lambda^*\mu^* 2^{m-1} + 3\mathcal{BA}\mu^*\lambda^* 2^n + 2(\mathcal{B})^2(\mu^*)^2$$

3. Conclusion

In this paper, we have introduced new integer sequences, named the Dual Hybrid Mersenne and Mersenne-Lucas sequences. We obtained the Binet's formulas and generating functions for these sequences. Also verified the above sequences through some well-known identities and a few relationships among them are presented.

4. References

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